PROBABILITY DISTRIBUTION

7.1 INTRODUCTION

In Semester - II, you have learnt the concept of probability and some basic theorems on probability.

In this present chapter, we will discuss some standard probability distributions. When the number of data is large and the probability of an event is very small, the basic theorems on probability do not work adequately. In such situations, probability distributions work well. Depending upon the nature of the data, various distributions are developed by Mathematicians from time to time.

7.2 PROBABILITY DISTRIBUTION OF RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a **random variable** or simply a **variate.** For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the values 2, 3, 4, ... 12, depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number 'a' occurs, then the corresponding random variable X is said to assume the value 'a' and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \leq C$ is written as $P(X \leq C)$.

If a random variable takes a finite set of values, it is called a discrete variate. On the other hand, if it assumes an infinite number of uncountable values, it is called a continuous variate.

If a random variable X takes the values $X_1, X_2, X_3, \ldots, X_n$ with probabilities $P_1, P_2, P_3, \ldots, P_n$ respectively where $P(X_1) = P_1, P(X_2) = P_2, \ldots$ and $P_1 + P_2 + P_3 + \ldots + P_n = 1$, then the probability distribution of the random variable X is written as below.

X :	X_1	X_2	X_3	...	X_n
P (X) :	P_1	P_2	P_3	...	P_n

7.2.1 Mean of Random Variable is

$$\mu = \frac{\Sigma P_i X_i}{\Sigma P_i}$$

Here $\Sigma P_i = P_1 + P_2 + P_3 + \ldots + P_n = 1$

$\therefore \quad \mu = \Sigma P_i X_i$

7.2.2 Variance of Random Variable is

$\sigma^2 = \Sigma P_i X_i - \mu^2$ where σ = S.D.

7.3 BINOMIAL DISTRIBUTION

If we perform a series of independent trials such that for each trial 'p' is the probability of a success and 'q' is that of failure, then the probability of r successes in a series of n trials is given by ${}^nC_r\, p^r\, q^{n-r}$, where r takes any integral value from 0 to n. The probabilities of 0, 1, 2, ..., r, ... n successes are, therefore, given by $q^n, {}^nC_1\, p \cdot q^{n-1}, {}^nC_2\, p^2 \cdot q^{n-2}, \ldots, {}^nC_r\, p^r\, q^{n-r}, \ldots, p^n$.

The probability of the number of successes so obtained is called the binomial distribution for the simple reason that the probabilities are the successive terms in the expansion of the binomial $(q+p)^n$.

$\therefore$ The sum of the probabilities

$$= q^n + {}^nC_1\, p \cdot q^{n-1} + {}^nC_2\, p^2 \cdot q^{n-2} + \ldots + p^n$$

$$= (q+p)^n = 1$$

Notes :

(1) Binomial distribution is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

(2) Mean of binomial distribution $= np$

$\therefore$ Mean $= \mu = n \cdot p$

(3) Standard deviation, $\sigma = \sqrt{n \cdot p \cdot q}$

(4) Variance, $V = npq$

The following illustrative examples are based on Binomial distribution :

●●● ILLUSTRATIVE EXAMPLES ●●●

(1) An unbiased coin is tossed 5 times. Find the probability of getting

(a) three heads (b) at least 4 heads.

Solution : In this case,

the number of independent trials, $n = 5$

In each trial, the probability of getting heads is $\frac{1}{2}$

$\therefore \; p = \frac{1}{2}, \; q = \frac{1}{2}$ ... {$\because q = 1 - p$)

By the binomial distribution, we have

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{n-r}$$

(a) When exactly 3 heads are obtained, $r = 3$

$$\therefore \; P(3) = {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3}$$

$$= {}^5C_2 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 \qquad \ldots {}^nC_r = {}^nC_{n-r}$$

$$= \frac{5 \times 4}{1 \times 2} \times \left(\frac{1}{2}\right)^5$$

$$= 10 \times \frac{1}{32} = \frac{5}{16}$$

(b) When at least 4 heads are obtained i.e. $r = 4$ or 5

$$\therefore \; P(r = 4 \text{ or } r = 5) = P(r = 4) + P(r = 5)$$

$$= {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{5-4} + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{5-5}$$

$$= {}^5C_1 \cdot \left(\frac{1}{2}\right)^5 + {}^5C_0 \cdot \left(\frac{1}{2}\right)^5$$

$$= [5 + 1] \cdot \left(\frac{1}{2}\right)^5$$

$$= 6 \times \frac{1}{32} = \frac{3}{16}$$

(2) The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that

(a) exactly two will be defective

(b) at least two will be defective

(c) none will be defective.

Solution : In this case, $n = 12$

p = probability that a pen manufactured will be defective

$= \frac{1}{10}$ or 0.1

$\therefore q = 1 - p = 1 - 0.1 = 0.9$ = probability of non-defective pen

By binomial distribution, we have

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{n-r}$$

(a) Exactly two will be defective i.e. $r = 2$

$$\therefore P(2) = {}^{12}C_2 \cdot (0.1)^2 \cdot (0.9)^{12-2}$$

$$= \frac{12 \times 11}{1 \times 2} \times (0.1)^2 \cdot (0.9)^{10}$$

$$= 6 \times 11 \times (0.01)(0.3487)$$

$$= 0.2301$$

(b) At least two will be defective

$= 1 - P$ (either none or one is non-defective)

$$= 1 - [P(r = 0) + P(r = 1)]$$

$$= 1 - \left[{}^{12}C_0 \cdot (0.9)^{12} + {}^{12}C_1 \cdot (0.1)^1 \cdot (0.9)^{11}\right]$$

$$= 1 - [0.2824 + (12)(0.1)(0.3138)]$$

$$= 1 - [0.2824 + 0.3766]$$

$$= 1 - [0.659]$$

$$= 0.341$$

(c) None will be defective i.e. $r = 0$

$$P(r = 0) = {}^{12}C_0 \cdot (0.1)^0 \cdot (0.9)^{12-0}$$

$$= 1 \times (0.9)^{12}$$

$$= 0.2824$$

(3) In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain atleast 3 defective parts.

Solution : This is based on binomial distribution.

Here $n = 20$, mean $= 2$

But for binomial distribution, mean $= n \cdot p$

$$\therefore 2 = 20 \times p$$

$$\therefore p = \frac{2}{20} = \frac{1}{10} \quad 0.1$$

$\therefore q$ = the probability of a non-defective part $= 1 - p$

$$= 1 - 0.1$$

$$= 0.9$$

The probability of at least three defectives in a sample of 20

= 1 − [Probability that either none, or one, or two are non-defective parts]

$$= 1 - \left[{}^{20}C_0 \cdot (0.9)^{20} + {}^{20}C_1 \cdot (0.1) \cdot (0.9)^{19} + {}^{20}C_2 \cdot (0.1)^2 \cdot (0.9)^{18}\right]$$

$$= 1 - \left[1 \times (0.9)^2 + 20\,(0.1)\,(0.9) + \frac{20 \times 19}{1 \times 2} \times (0.1)^2\right](0.9)^{18}$$

$$= 1 - [0.81 + 1.8 + 1.9]\,(0.9)^{18}$$

$$= 1 - 4.51 \times (0.9)^{18}$$

$$= 1 - 0.677 = 0.323$$

Thus, the number of samples having at least three defective parts out of 1000 samples.

$$= 1000 \times 0.323 = 323$$

(4) If 30% of the bulbs produced are defective, find the probability that out of 4 bulbs selected

(a) one is defective

(b) at the most two are defective.

Solution : 30% bulbs are defective

$\therefore$ p = probability of getting defective bulb $= \frac{30}{100} = \frac{3}{10}$

$\therefore$ q = probability of non-defective bulb $= 1 - p = 1 - \frac{3}{10} = \frac{7}{10}$

Four bulbs are selected $\therefore$ $n = 4$

By binomial distribution, we have

$$P(x = r) = {}^nC_r \cdot p^r \cdot q^{n-r}$$

(a) Probability that one bulb is defective i.e. $r = 1$

$$P(1) = {}^4C_1 \left(\frac{3}{10}\right)^1 \cdot \left(\frac{7}{10}\right)^{4-1}$$

$$= 4 \times \frac{3}{10} \times \left(\frac{7}{10}\right)^3$$

$$= 4 \times 0.3 \times 0.343$$

$$= 0.4116$$

(b) Probability that at the most two bulbs are defective i.e. $r = 0, 1, 2$

$\therefore$ $P(0 \text{ or } 1 \text{ or } 2) = P(0) + P(1) + P(2)$

$$= {}^4C_0 \left(\frac{7}{10}\right)^4 + {}^4C_1 \left(\frac{3}{10}\right) \cdot \left(\frac{7}{10}\right)^3 + {}^4C_2 \left(\frac{3}{10}\right)^2 \cdot \left(\frac{7}{10}\right)^2$$

using ${}^nC_0 = 1,\ {}^nC_1 = n,\ {}^nC_2 = \frac{n\,(n-1)}{1 \times 2}$

$$= \left[\left(\frac{7}{10}\right)^2 + 4\,(0.3)\,(0.7) + \frac{4 \times 3}{1 \times 2} \times (0.09)\right]\left(\frac{7}{10}\right)^2$$

$$= [0.49 + 0.84 + 0.54] \times 0.49$$

$$= 1.87 \times 0.49$$

$$= 0.9163$$

(5) A set of 5 coins was tossed 250 times and the frequencies of throws observed were as follows :

(6) **In 10 independent throws of a defective die, the probability that an even number will appear five times is twice the probability that an even number will appear four times. Find the probability that an even number will not appear at all in 10 independent throws of the die.**

Solution : In this case,

p = probability of getting an even number (success) in a single trial

q = probability of not getting an even number (failure)

n = number of throws = 10

Applying the binomial distribution, we have

$$P(x = r) = {}^{n}C_{r} \cdot p^{r} \cdot q^{n-r}$$

It is given that $P(r = 5) = 2 \times P(r = 4)$

$$\therefore \quad {}^{10}C_{5} \cdot p^{5} \cdot q^{5} = 2 \times {}^{10}C_{4} \cdot p^{4} \cdot q^{6}$$

$$\therefore \quad \frac{10\,!}{5\,! \times 5\,!}\, p^{5} \cdot q^{5} = 2 \times \frac{10\,!}{4\,! \times 6\,!}\, p^{4} \cdot q^{6}$$

$$\therefore \quad \frac{1}{5}\, p = \frac{2}{6}\, q$$

$$\therefore \quad \frac{p}{5} = \frac{q}{3}$$

$$\therefore \quad 3p = 5q$$

$$\therefore \quad 3p = 5(1 - p)$$

$$\therefore \quad 3p = 5 - 5p$$

$$\therefore \quad 8p = 5$$

$$\therefore \quad p = \frac{5}{8}$$

$$\therefore \quad q = 1 - p = 1 - \frac{5}{8} = \frac{3}{8}$$

Hence the probability that an even number will not appear at all (zero success) is :

$$= P(r = 0) = {}^{10}C_{0} \cdot q^{0} \cdot p^{10}$$

$$= 1 \times 1 \times \left(\frac{3}{8}\right)^{10} = \left(\frac{3}{8}\right)^{10}$$

7.4 POISSON DISTRIBUTION

In binomial distribution if we are given n and p, we can form probability distribution. But if 'n' is large and p or q are very small, the calculations become very difficult. In such situations, Poisson's distribution is applicable.

Thus, Poisson's distribution is a limiting case of binomial distribution when $n \to \infty$ and $p \to 0$ and is given by the formula :

$$P(x = r) = \frac{e^{-m} \cdot m^r}{r!} \qquad \dots (1)$$

where 'm' is mean and $m = n \cdot p$

The probability distribution determined by function (1) above is called the Poisson distribution and 'm' is called the **parameter** of the distribution. Evidently, this distribution is an infinite discrete probability distribution. The random variable 'r' associated with a Poisson distribution is called a **Poisson variate.**

7.4.1 Mean, $m = n \cdot p$

7.4.2 Variance, $V = m$

7.4.3 S.D $= \sqrt{m}$

The following examples illustrate the Poisson distribution :

●●● ILLUSTRATIVE EXAMPLES ●●●

(1) If a random variable has a Poisson distribution such that P (3) = P (4), find P (0) and P (1).

Solution : The probability function of Poisson distribution is given by

$$P(r) = \frac{m^r \cdot e^{-m}}{r\,!}$$

$$\therefore \quad P(3) = \frac{m^3 \cdot e^{-m}}{3\,!}$$

$$\text{and } P(4) = \frac{m^4 \cdot e^{-m}}{4\,!}$$

$$\text{But } P(3) = P(4)$$

$$\frac{m^3 \cdot e^{-m}}{3\,!} = \frac{m^4 \cdot e^{-m}}{4\,!}$$

$$\therefore \quad \frac{1}{3\,!} = \frac{m}{3\,! \times 4} \qquad \therefore \quad m = 4$$

Now (i) $P(0) = \frac{m^0 \cdot e^{-m}}{0\,!} = e^{-4} = 0.01832$

(ii) $P(1) = \frac{m^1 \cdot e^{-m}}{1\,!} = \frac{4 \times e^{-4}}{1} = 4 \times .01832 = 0.0733$

(2) Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.

Solution : In this problem,

p = probability of an ace of spade in a single trial $= \frac{1}{52}$

$n = 104$ such that $m = np$

$$\therefore \quad m = 104 \times \frac{1}{52} = 2$$

Next, according to Poisson's distribution function, we get

$$P(r) = \frac{m^r \cdot e^{-m}}{r\,!}$$

and P (at least one in 104 consecutive trials)

$$= 1 - P(r = 0)$$

$$= 1 - \left[\frac{m^0 \cdot e^{-m}}{0\,!}\right] \qquad \text{... Note this step carefully}$$

$$= 1 - e^{-m} \qquad \text{... } 0\,! = 1,\ m^0 = 1$$

$= 1 - e^{-2}$ $\quad \ldots m = 2$

$= 1 - 0.135$ $\quad e^{-2} = \frac{1}{e^2} = \frac{1}{(2.718)^2} = 0.135$

$= 0.865$

(3) **Fit a Poisson distribution to set of following observations :**

$$= \frac{}{r\,!}$$

(4) If the probability that an individual suffers a bad reaction from injection is 0.001. Determine the probability that out of 1000 individuals

(a) **exactly 2**

(b) **at least 2**

(c) **at most 2**

(d) **more than 2 will suffer a bad reaction.**

Solution : In this problem,

$$n = 1000, \quad p = 0.001$$

$$\therefore \text{ Mean, } m = n \cdot p = 1000 \times 0.001 = 1$$

Then, according to Poisson's distribution function, we have

$$P(r) = \frac{m^r \cdot e^{-m}}{r\,!} \qquad \ldots (1)$$

(a) Exactly 2 will suffer a bad reaction i.e. $r = 2$

$$\therefore P(r = 2) = \frac{(1)^2 \cdot e^{-1}}{2\,!} = \frac{1}{2} \cdot \frac{1}{e} = \frac{1}{2\,(2.718)} = \frac{1}{5.436} = 0.1840$$

(b) At least 2 will suffer a bad reaction i.e. r = 2, 3, ..., 1000

$\therefore$ P (r = at least two) $= P(2) + P(3) + \ldots + P(1000)$

$$= 1 - [P(0) + P(1)]$$

$$= 1 - \left[\frac{(1)^0 \cdot e^{-1}}{0\,!} + \frac{(1)^1 \cdot e^{-1}}{1\,!}\right]$$

$$= 1 - [1 + 1] \cdot e^{-1}$$

$$= 1 - \frac{2}{e}$$

$$= 1 - \frac{2}{2.718} = 1 - 0.736 = 0.2640$$

(c) At most two will suffer a bad reaction i.e. r = 0, 1, 2

$\therefore$ P (r = at most two) $= P(0) + P(1) + P(2)$

$$= \frac{(1)^0 \cdot e^{-1}}{0\,!} + \frac{(1)^1 \cdot e^{-1}}{1\,!} + \frac{(1)^2 \cdot e^{-1}}{2\,!}$$

$$= e^{-1} + e^{-1} + \frac{e^{-1}}{2}$$

$$= \left[1 + 1 + \frac{1}{2}\right] \cdot e^{-1}$$

$$= \frac{5}{2} \cdot \frac{1}{e} = \frac{5}{2\,(2.718)} = \frac{5}{5.436} = 0.92$$

(d) More than two will suffer a bad reaction i.e. r = 3, 4, ...

$\therefore$ P (r = more than two) $= 1 - [P(0) + P(1) + P(2)]$

$= 1 - 0.92$ ... from case (c)

$= 0.08$

(5) If 5% of the electric bulbs manufactured by a company are defective, use Poisson distribution to find the probability that in a sample of 100 bulbs

(a) none is defective and

(b) five bulbs are defective. (Given $e^{-5} = 0.007$)

Solution : In this problem,

p = probability of defective bulbs

$= 5\% = \frac{5}{100} = 0.05$ (small)

n = 100 (large)

Poisson's distribution is most suited.

$\therefore\ m = n \cdot p = 100 \times 0.05 = 5$

According to Poisson's distribution function

$$P(r) = \frac{m^r \cdot e^{-m}}{r\,!}$$

(a) None is defective i.e. r = 0

$$\therefore\ P(0) = \frac{(5)^0 \cdot e^{-5}}{0\,!} = e^{-5} = 0.007$$

(b) Five bulbs are defective i.e. r = 5

$$\therefore \quad P(5) = \frac{(5)^5 \cdot e^{-5}}{5\,!}$$

$$= \frac{3125 \times 0.007}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= \frac{21.875}{120}$$

$$= 0.1823$$