Tutorial 6

Q1

① $y = \sin^{-1}\left(\frac{1-4x^2}{1+4x^2}\right)$

~~let~~ $y = \sin^{-1}\left(\frac{1-(2x)^2}{1+(2x)^2}\right)$

put, $2x = \tan\theta$, $\theta = \tan^{-1}(2x)$

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$y = \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2(\tan^{-1}(2x))$$

differentiate y w.r.t. to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\tan^{-1}(2x)\right)$$

$$= 0 - 2\frac{d}{dx}\left(\tan^{-1}(2x)\right)$$

$$= -2\left(\frac{1}{1+(2x)^2}\right)\frac{d}{dx}(2x)$$

$$= \frac{-2}{1+4x^2}(2)$$

$$\frac{dy}{dx} = \frac{-4}{1+4x^2}$$

(2) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

putting $x = \tan\theta$, $\theta = \tan^{-1}(x)$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$y = \sin^{-1}(\sin 2\theta)$$

$$y = 2\theta$$

$$y = 2\tan^{-1}x$$

differentiate w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

(3) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

putting $x = \tan\theta$, $\theta = \tan^{-1}(x)$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$y = 2\tan^{-1}(x)$$

differentiate w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}(x))$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

④ $y = \tan^{-1}\left(\frac{2x}{1+35x^2}\right)$

$= \tan^{-1}\left(\frac{7x-5x}{1+(7x)(5x)}\right)$

$y = \tan^{-1}(7x) - \tan^{-1}(5x)$

differentiate w.r.to x

$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(7x)) - \frac{d}{dx}(\tan^{-1}(5x))$

$= \frac{1}{1+(7x)^2}\frac{d}{dx}(7x) - \frac{1}{1+(5x)^2}\frac{d}{dx}(5x)$

$= \frac{7}{1+49x^2} - \frac{5}{1+25x^2}$

⑤ $y = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

putting $x = \tan\theta$, $\theta = \tan^{-1}(x)$

$y = \sin^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)$

$= \sin^{-1}\left(\frac{1}{\sqrt{\sec^2\theta}}\right)$

$= \sin^{-1}\left(\frac{1}{\sec\theta}\right)$

$= \sin^{-1}(\cos\theta)$

$= \sin^{-1}(\sin(\pi/2 - \theta))$

$= \frac{\pi}{2} - \theta$

$y = \frac{\pi}{2} - \tan^{-1}x$

differentiate w.r.to x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}x\right)$

$= 0 - \left(\frac{1}{1+x^2}\right)$

$\frac{dy}{dx} = \frac{-1}{1+x^2}$

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(6) $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

~~= tan~~

putting $x = \tan\theta$, $\theta = \tan^{-1}(x)$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta}{\tan\theta} - \frac{1}{\tan\theta}\right)$$

$$= \tan^{-1}(\operatorname{cosec}\theta - \cot\theta)$$

$$= \tan^{-1}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta} \times \frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}\right)$$

$$= \tan^{-1}\left(\frac{1-1+\sin^2\theta}{\sin\theta(1+\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin(\theta/2)\cos(\theta/2)}{1-\sin^2(\theta/2)+\cos^2(\theta/2)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin(\theta/2)\cos(\theta/2)}{\cos^2(\theta/2)+\cos^2(\theta/2)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}\right)$$

$$= \tan^{-1}\left(\frac{\sin(\theta/2)}{\cos(\theta/2)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)$$

$$y = \frac{\theta}{2} = \frac{\tan^{-1}(x)}{2}$$

differentiate w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\tan^{-1}(x)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2+2x^2}$$

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(7) $y = \sin^{-1}\left(\frac{\cos x + \sin x}{\sqrt{2}}\right)$

Let $= \frac{(\cos x + \sin x)}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x$

$= \sin\left(\frac{\pi}{4}\right)\cos x + \cos\left(\frac{\pi}{4}\right)\sin x$

$= \sin\left(\frac{\pi}{4} + x\right)$

substitute in y

$\therefore y = \sin^{-1}\left(\sin\left(\frac{\pi}{4} - x\right)\right)$

$y = \left(\frac{\pi}{4} - x\right)$

differentiate w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$

$= 0 - 1$

$\frac{dy}{dx} = 1$

(8) $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$\cancel{= \tan^{-1}\left(\frac{\cos x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\right)}$

$y = \tan^{-1}\left(\frac{\sin(\pi/2 - x)}{1+\cos(\pi/2 - x)}\right)$

$\cancel{y =}$ Let $(\pi/2 - x) = A$

$\therefore y = \tan^{-1}\left(\frac{\sin A}{1+\cos A}\right)$

$= \tan^{-1}\left(\frac{2\sin(A/2)\cos(A/2)}{2\sin^2(A/2)}\right)$

$= \tan^{-1}(\cot(A/2))$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{A}{2}\right)\right)$

$y = \left(\frac{\pi}{2} - \frac{A}{2}\right)$

resubstituting for A

$$y = \frac{\pi}{2} - \frac{\left(\frac{\pi}{2} - x\right)}{2}$$

$$= \frac{\frac{2\pi}{2} - \frac{\pi}{2} + x}{2}$$

$$= \frac{\pi - \frac{\pi}{2} + x}{2}$$

$$= \frac{\frac{\pi}{2} + x}{2}$$

$$= \frac{\frac{\pi}{2}}{2} + \frac{x}{2}$$

$$y = \pi + \frac{x}{2}$$

differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\pi + \frac{x}{2}\right)$$

$$= 0 + \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}$$

(9) $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

putting $x = \tan\theta$, $\theta = \tan^{-1}x$

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}(\cos(\pi/2 - 2\theta)$$

$$= \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2\tan^{-1}x$$

differentiate w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(2\tan^{-1}x)$$

$$= 0 - 2\left(\frac{1}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{-2}{1+x^2}$$

10 $y = \cos^{-1}(1-2\sin^2 x)$

$y = \cos^{-1}(\cos 2x)$

$y = 2x$

differentiate w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx} 2x$

$\frac{dy}{dx} = 2$

Q2 If $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$, $\frac{dy}{dx} = ?$

sol. Given: $x = 3\cos\theta - 2\cos^3\theta$

differentiate w.r.to θ

$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$

$= 3(-\sin\theta) - 2(3\cos^2\theta)\frac{d}{d\theta}(\cos\theta)$

$= -3\sin\theta - 6\cos^2\theta(-\sin\theta)$

$= -3\sin\theta + 6\cos^2\theta\sin\theta$

$\frac{dx}{d\theta} = 3\sin\theta(2\cos^2\theta - 1) \rightarrow (1)$

Given: $y = 3\sin\theta - 2\sin^3\theta$

differentiate w.r.to θ

$\frac{dy}{d\theta} = \frac{d}{d\theta}(3\sin\theta) - \frac{d}{d\theta}(2\sin^3\theta)$

$= 3\cos\theta - 2(3\sin^2\theta)\frac{d}{d\theta}(\sin\theta)$

$= 3\cos\theta - 6\sin^2\theta\cos\theta$

$\frac{dy}{d\theta} = 3\cos\theta(1 - 2\sin^2\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3\cos\theta(1-2\sin^2\theta)}{3\sin\theta(2\cos^2\theta-1)}$$

$$= \tan\theta\left(\frac{\cos 2\theta}{\cos 2\theta}\right)$$

$$\frac{dy}{dx} = \cancel{\tan\theta}\ \cot\theta$$

Q3 If $x = a(2\alpha - \sin 2\alpha)$, $y = a(1-\cos 2\alpha)$; $\frac{dy}{dx} = ?$

Sol. Given: $x = a(2\alpha - \sin 2\alpha)$

differentiate w.r.t α

$$\frac{dx}{d\alpha} = a\left(\frac{d}{d\alpha}(2\alpha) - \frac{d}{d\alpha}(\sin 2\alpha)\right)$$

$$= a\left(2 - \cos 2\alpha \frac{d}{d\alpha}(2\alpha)\right)$$

$$= a\left(2 - \cos 2\alpha(2)\right)$$

$$= a\left(2(1-\cos 2\alpha)\right)$$

$$= 2a(1-\cos 2\alpha)$$

$$\frac{dx}{d\alpha} = 2a(\sin 2\alpha)$$

Given: $y = a(1-\cos 2\alpha)$

differentiate w.r.to α

$$\frac{dy}{d\alpha} = a\left(\frac{d}{d\alpha}(1) - \frac{d}{d\alpha}(\cos 2\alpha)\right)$$

$$= a\left((0) - (-\sin 2\alpha)\frac{d}{d\alpha}(2\alpha)\right)$$

$$= a(2\sin 2\alpha)$$

$$\frac{dy}{d\alpha} = 2a\sin 2\alpha$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}} = \frac{2a\sin 2\alpha}{2a\sin 2\alpha} = 1$$

Q4 If $x = a\cos^3\theta$, $y = b\sin^3\theta$, $\frac{dy}{dx} = ?$

Sol. $x = a\cos^3\theta$ — (given)

differentiate w.r.t. to θ

$$\frac{dx}{d\theta} = a\left(\frac{d}{d\theta}(\cos^3\theta)\right)$$

$$= a\left(3\cos^2\theta \frac{d}{d\theta}(\cos\theta)\right)$$

$$= 3a\cos^2\theta(-\sin\theta)\}$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$y = b\sin^3\theta$ — (given)

differentiate w.r.t. to θ

$$\frac{dy}{d\theta} = b\left(\frac{d}{d\theta}(\sin^3\theta)\right)$$

$$= b\left(3\sin^2\theta \frac{d}{d\theta}(\sin\theta)\right)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = -\frac{b}{a}\tan\theta$$

Q5 $x = \frac{3at}{1+t^2}$, $y = \frac{3a}{1+t^2}$, $\frac{dy}{dx} = ?$ Q6

Sol. Given: $x = \frac{3at}{1+t^2}$

differentiate w.r.to t

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{3at}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(3at) - (3at)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(3a) - 3at(2t)}{(1+t^2)^2}$$

$$= \frac{3a(1+t^2-2t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

Given: $y = \frac{3a}{1+t^2}$, differentiate w.r. to t

$$\frac{dy}{dt} = (1+t^2)\frac{d}{dt}(3a) - 3a\frac{d}{dt}(1+t^2) / (1+t^2)^2$$

$$= (1+t^2)(3a) - 3a(2t) / (1+t^2)^2$$

$$\frac{dy}{dt} = 3a(1+t^2-2t) / (1+t^2)^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{3a(1+t^2-2t)}{(1+t^2)^2}}{\frac{3a(1-t^2)}{(1+t^2)^2}}$$

$$= \frac{1+t^2-2t}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2} = \left(\frac{1-t}{1+t}\right)$$

Q6 $y = e^{\sin 2\alpha}, \quad x = e^{\cos 2\alpha}$

given $y = e^{\sin 2\alpha}$, differentiate w.r.to α

$$\frac{dy}{d\alpha} = \frac{d}{d\alpha}\left(e^{\sin 2\alpha}\right)$$

$$= e^{\sin 2\alpha} \frac{d}{d\alpha}(\sin 2\alpha)$$

$$= e^{\sin 2\alpha}(\cos 2\alpha)(2)$$

$$\frac{dy}{d\alpha} = 2\cos 2\alpha \, e^{\sin 2\alpha}$$

given $x = \cancel{\cos}\, e^{\cos 2\alpha}$, differentiate w.r.to α

$$\frac{dx}{d\alpha} = \frac{d}{d\alpha}\left(e^{\cos 2\alpha}\right)$$

$$= e^{\cos 2\alpha} \frac{d}{d\alpha}(\cos 2\alpha)$$

$$= e^{\cos 2\alpha}(-\sin 2\alpha)(2)$$

$$\frac{dx}{d\alpha} = -2\sin 2\alpha \, e^{\cos 2\alpha}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}}$$

$$= \frac{\cancel{2}\cos 2\alpha \, e^{\sin 2\alpha}}{-\cancel{2}\sin 2\alpha \, e^{\cos 2\alpha}} \rightarrow (1)$$

But $\log x = \log\left(e^{\cos 2\alpha}\right) = \cos 2\alpha(\log(e))$

$\log y = \log\left(e^{\sin 2\alpha}\right) = \sin 2\alpha(\log(e))$

$$\frac{\log x}{\log y} = \frac{\cos 2\alpha}{\sin 2\alpha} \rightarrow$$

substituting in (1)

$$\frac{dy}{dx} = -\frac{y \log x}{x \log y}$$

Q7 If $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$, $\frac{dy}{dx} = ?$

Given: $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$

putting $t = \tan\theta$, $\theta = \tan^{-1}(t)$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta = 2\tan^{-1}(t)$$

differentiate w.r. to t

$$\frac{dy}{dt} = 2\frac{d}{dt}(\tan^{-1}(t))$$

$$= 2\left(\frac{1}{1+t^2}\right)$$

$$\frac{dy}{dt} = \frac{2}{1+t^2}$$

Given: $x = \tan^{-1}\left(\frac{2t}{1+t^2}\right)$

putting $t = \tan\theta$, $\theta = \tan^{-1}(t)$

$$x = \tan^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$x = 2\theta = 2\tan^{-1}(t)$$

differentiate w.r. to t

$$\frac{dx}{dt} = 2\frac{d}{dt}(\tan^{-1}(t))$$

$$= 2\left(\frac{1}{1+t^2}\right)$$

$$= \frac{2}{1+t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$$

Q8 Differentiate $\log(1+x^2)$ w.r.t $\tan^{-1}(x)$

Let $y = \log(1+x^2)$

differentiate y w.r.to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log(1+x^2)\right)$$

$$= \frac{1}{1+x^2}\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{1+x^2}(2x)$$

$$\frac{dy}{dx} = \frac{2x}{1+x^2}$$

Let $z = \tan^{-1}(x)$

differentiate z w.r.to z

$$\frac{dz}{dx} = \frac{d}{dx}(\tan^{-1}x)$$

$$\frac{dz}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$= \frac{\frac{2x}{(1+x^2)}}{\frac{1}{(1+x^2)}} = \frac{2x}{1+x^2} \times \frac{(1+x^2)}{1} = 2x$$

$$\therefore \boxed{\frac{dy}{dz} = 2x}$$

Q9 Differentiate $(\log x)^2$ w.r.t 5^x

Let $y = (\log x)^2$, differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}((\log x)^2)$$

$$= 2\log x \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{2\log x}{x}$$

Let $z = 5^x$, differentiate w.r.t x

$$\frac{dz}{dx} = \frac{d}{dx}(5^x)$$

$$= 5^x \log 5$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$= \frac{\frac{2\log x}{x}}{5^x \log 5}$$

$$\frac{dy}{dz} = \frac{2\log x}{x 5^x \log 5}.$$

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Q10 diff. $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ w.r.t $\sin^{-1}x$

let $y = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

let $x = \sin\theta$, $\theta = \sin^{-1}x$

$y = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$

$= \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$

$= \sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$= \sec^{-1}(\sec\theta)$

$= \theta$

$y = \sin^{-1}x$

differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)$

$= \frac{1}{\sqrt{1-x^2}}$

let $z = \sin^{-1}x$, differentiate w.r.t x

$\frac{dz}{dx} = \frac{d}{dx}(\sin^{-1}x)$

$= \frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$

$= \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$

$\frac{dy}{dz} = 1$

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Q11 Diff. $\log(x\sin x)$ w.r.t $\frac{1}{x}$ — Q12

let $y = \cancel{d} \log(x\sin x)$, differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}(\log(x\sin x))$

$= \frac{1}{x\sin x} \cancel{\frac{d}{dx}} \left(x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x)\right)$

$= \frac{1}{x\sin x}\left(x\cos x + \sin x(1)\right)$

$= \frac{x\cos x + \sin x}{x\sin x}$

$= \frac{x\cos}{x\sin x} + \frac{\sin x}{x\sin x} = \cancel{x}$

$\frac{dy}{dx} = x\cot x + \frac{1}{x}$

let $z = \frac{1}{x}$, differentiate w.r.t x

$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{-1}{x^2}$

$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$

$= \frac{x\cot x + \frac{1}{x}}{\frac{-1}{x^2}}$

$= \left(x\cot x + \frac{1}{x}\right) \times \left(\frac{x^2}{-1}\right)$

$\frac{dy}{dz} = -x^2\left(x\cot x + \frac{1}{x}\right)$

Q12 Diff $e^{3x}\sin 2x$ w.r.t x^2

let $y = e^{3x}\sin 2x$, differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(e^{3x}\sin 2x)$$

$$= e^{3x}\frac{d}{dx}(\sin 2x) + \sin 2x \frac{d}{dx}(e^{3x})$$

$$= e^{3x}\cos 2x \cdot \frac{d}{dx}(2x) + \sin 2x\,(e^{3x})\frac{d}{dx}(3x)$$

$$= 2e^{3x}\cos 2x + 3\sin 2x\, e^{3x}$$

$$\frac{dy}{dx} = e^{3x}(2\cos 2x + 3\sin 2x)$$

let $z = x^2$ differentiate w.r.t x

$$\frac{dz}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{dz}{dx} = 2x$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$\frac{dy}{dz} = \frac{e^{3x}(2\cos 2x + 3\sin 2x)}{2x}$$

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Q13 $y = e^{m\sin^{-1}x}$: prove $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.

Given: $y = e^{m\sin^{-1}x} \rightarrow (1)$

differentiate w.r.t x

$$\frac{dy}{dx} = e^{m\sin^{-1}x}\frac{d}{dx}(m\sin^{-1}x)$$

$$= e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right)$$

$$= \frac{me^{m\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{my}{\sqrt{1-x^2}} \rightarrow (\text{from } (1)) \rightarrow (2)$$

$$(\sqrt{1-x^2})\frac{dy}{dx} = my$$

diff. w.r.t x

$$(\sqrt{1-x^2})\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}(\sqrt{1-x^2}) = m\frac{d}{dx}y$$

$$(\sqrt{1-x^2})\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)\right) = m\frac{dy}{dx}$$

$$(\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx} = m\frac{dy}{dx}$$

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = m\left(\frac{my}{\sqrt{1-x^2}}\right) - (\text{From } (2))$$

$$(\sqrt{1-x^2})\left(\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}\right) = m^2y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Hence proved.

Q14

Q14 $y = \cos(m\sin^{-1}x)$: prove $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

$y = \cos(m\sin^{-1}x) \rightarrow (1)$

differentiate y w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}(\cos(m\sin^{-1}x))$

$= -\sin(m\sin^{-1}x)\frac{d}{dx}(m\sin^{-1}x)$

$= -\sin(m\sin^{-1}x)\, m\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = -\frac{m\sin(m\sin^{-1}x)}{\sqrt{1-x^2}} \rightarrow (2)$

Again differentiate w.r.t x

$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d}{dx}(\sqrt{1-x^2}) = -m\frac{d}{dx}\cos(m\sin^{-1}x)$

$\frac{d}{dx}(m\sin^{-1}x)$

$\therefore \sqrt{1-x^2}\frac{d^2y}{dx^2} + \left(\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)\right)\frac{dy}{dx} = -m\cos(m\sin^{-1})\frac{m}{\sqrt{1-x^2}}$

$\therefore \sqrt{1-x^2}\frac{d^2y}{dx^2} + \left(\frac{2x}{2\sqrt{1-x^2}}\right)\frac{dy}{dx} = -\frac{m^2\cos(m\sin^{-1})}{\sqrt{1-x^2}}$

$(\sqrt{1-x^2})\left(\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}\right) = -m^2y$ (from (1))

$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -m^2y$

$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + m^2y = 0$

Q15 $y = \log(x + \sqrt{x^2+1}) \rightarrow (1)$

differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \frac{d}{dx}(x + \sqrt{x^2+1})$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{x^2+1})\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2\sqrt{x^2+1}} \frac{d}{dx}(x^2+1)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2\sqrt{x^2+1}}(2x)\right)$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}}\left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \rightarrow (2)$$

$$\sqrt{x^2+1}\frac{dy}{dx} = 1$$

Again differentiate w.r.t x

$$\sqrt{x^2+1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{d}{dx}(\sqrt{x^2+1})\right) = 0$$

$$\sqrt{x^2+1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)\right) = 0$$

$$\sqrt{x^2+1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{2\sqrt{x^2+1}}(2x)\right) = 0$$

$$\sqrt{x^2+1}\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2+1}}\frac{dy}{dx} = 0$$

$$\frac{(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}}{\sqrt{x^2+1}} = 0$$

$$x^2+1\frac{d^2y}{dx} + x\frac{dy}{dx} = 0$$

Q16 If $y = Ae^x + Be^{-x}$, prove $\frac{d^2y}{dx^2} - y = 0$

$y = Ae^x + Be^{-x} \rightarrow (1)$

differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(Ae^x) + \frac{d}{dx}(Be^{-x})$$

$$= Ae^x + Be^{-x}\frac{d}{dx}(-x)$$

$$= Ae^x + Be^{-x}(-1)$$

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

Again differentiate w.r.t x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(Ae^x) - \frac{d}{dx}(Be^{-x})$$

$$= Ae^x - Be^{-x}\frac{d}{dx}(-x)$$

$$= Ae^x - Be^{-x}(-1)$$

$$= Ae^x + Be^{-x}$$

$$\frac{d^2y}{dx^2} = y \quad (\text{from } (1))$$

$$\frac{d^2y}{dx^2} - y = 0$$

Q17 $y = 2\cos(\log x) + 3\sin(\log x)$ prove $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$y = 2\cos(\log x) + 3\sin(\log x) \to ①$

differentiate w.r.t. to x

$$\frac{dy}{dx} = 2\frac{d}{dx}(\cos(\log x)) + 3\frac{d}{dx}(\sin(\log x))$$

$$= 2\left(-\sin(\log x)\frac{1}{x}\right) + 3\left(\cos(\log x)\frac{1}{x}\right)$$

$$= -\frac{2\sin\log x}{x} + \frac{3\cos(\log x)}{x}$$

$$\frac{dy}{dx} = \frac{-2\sin(\log x) + 3\cos(\log x)}{x}$$

$$x\frac{dy}{dx} = -2\sin(\log x) + 3\cos(\log x)$$

Again differentiate w.r.t. x

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d}{dx}(x) = \frac{d}{dx}(-2\sin(\log x) + 3\cos(\log x))$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -2\cos(\log x)\frac{1}{x} + 3(-\sin(\log x))\frac{1}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-2\cos(\log x) - 3(\sin(\log x))}{x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(2\cos(\log x) + 3(\sin(\log x)))$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \quad \text{(from ①)}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Q 18. $y = \log x$, prove: $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

$y = \log x \rightarrow (1)$

differentiate w.r.to x

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\log x)$

$\dfrac{dy}{dx} = \dfrac{1}{x} \rightarrow (2)$ $\therefore x\dfrac{dy}{dx} = 1$

Again differentiate w.r.to x

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}(1) = \dfrac{d}{dx}(1)$

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

Hence proved.