Date

# Tutorial 8

Q1

1 $I = \int \sin^3 \cos x \, dx \rightarrow (I)$

Let $u = \sin x$

differentiate $u$ w.r.t. to $x$

$du = \cos x \, dx$

substituting for $(I)$

$$I = \int u^3 \, du$$

$$= \frac{u^{3+1}}{3+1} + c$$

$$I = \frac{\sin^4}{4} + c$$

2 $I = \int \frac{(\tan^{-1} x)^2}{1+x^2} \, dx \rightarrow (1)$

Let $u = \tan^{-1} x$

differentiate $u$ w.r.t. to $x$

$$du = \frac{1}{1+x^2} \, dx$$

substituting for $(1)$

$$I = \int u^2 \, du$$

$$= \frac{u^{2+1}}{2+1} + c$$

$$= \frac{(\tan^{-1} x)^3}{3} + c$$

3 $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx \rightarrow (1)$

let $xe^x = u$

differentiate u w.r.to x

$$\frac{du}{dx} = \frac{d}{dx}(xe^x)$$

$$\frac{du}{dx} = e^x\left(\frac{d}{dx}(x)\right) + \cancel{e^x}\, x \frac{d}{dx} e^x$$

$$= e^x + xe^x$$

$$\frac{du}{dx} = e^x(1+x)$$

$$dx = \frac{du}{e^x(1+x)}$$

substituting in (1)

$$I = \int \frac{e^x(1+x)}{\cos^2(u)} \cdot \frac{du}{e^x(1+x)}$$

$$= \int \frac{du}{\cos^2(u)}$$

$$= \int \sec^2(u)\, du$$

$$= \tan(u) + C$$

$$I = \tan(xe^x) + C$$

4 $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx \rightarrow (1)$

$S2$

Let $u = \sqrt{x}$

differentiate u w.r.t. to x

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$2\,du = \frac{1}{\sqrt{x}}\,dx$$

substituting in (1)

$$I = \int \sin(u) \cdot 2\,du$$
$$= 2\int \sin(u)\,du$$
$$= 2(-\cos(u)) + C$$
$$= -2\cos(u) + C$$
$$I = -2\cos(\sqrt{x}) + C$$

5 $I = \int \frac{(2+\log x)^2}{x}\, dx \rightarrow (1)$

Let $u = 2 + \log x$

differentiate u w.r.t. to x

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

substitute in (1).

$$I = \int u^2\, du$$
$$= \frac{u^{2+1}}{2+1} + C$$
$$= \frac{u^3}{3} + C$$
$$= \frac{(2+\log x)^3}{3} + C$$

Q2

1 $I = \int \frac{dx}{x^2+4x+5}$

considers, $x^2+4x+5 = (x+2)^2 + (5-2^2)^2$

$\cancel{x^2+4x+5} = (x+2)^2+1$

$I = \int \frac{dx}{(x+2)^2+1} \rightarrow (1)$

Let $u = x+2$

differentiate $u$ w.r.t to $x$

$\frac{du}{dx} = 1$

$du = dx$

substituting in (1)

$I = \int \frac{du}{u^2+1}$

$= \cancel{\log|u^2} \frac{1}{2u}\int \frac{2u\,du}{u^2+1}$

$= \frac{1}{2u}(\log|u^2+1|) + C$

$= \frac{1}{2u}(\log|(x+2)^2+1|) + C$

$= \frac{1}{2x+4}(\log|(x+2)^2+1|) + C$

$= \tan^{-1}(u) + C$

$= \tan^{-1}(x+2) + C$

2 $I = \int \frac{dx}{2x^2+3x+2}$

consider, $2x^2+3x+2 = 2\left(x^2+\frac{3}{2}x+1\right)$

$$= 2\left(\left(x+\frac{3}{4}\right)^2+\frac{7}{16}\right)$$

$$= 2\left(x+\frac{3}{4}\right)^2+\frac{7}{8}$$

$$I = \int \frac{dx}{2(x+3/4)^2+7/8}$$

$$= \int \frac{dx}{\frac{7}{8}\left(\frac{16}{7}\left(x+\frac{3}{4}\right)^2+1\right)}$$

$$= \frac{8}{7}\int \frac{dx}{\left(\frac{4}{\sqrt{7}}\left(x+\frac{3}{4}\right)\right)^2+1}$$

Let $u = \frac{4}{\sqrt{7}}\left(x+\frac{3}{4}\right)$

differentiate $u$ w.r.t $x$

$$\frac{du}{dx} = \frac{4}{\sqrt{7}}$$

$$du = \frac{4}{\sqrt{7}}dx$$

$$dx = \frac{4}{\sqrt{7}}du$$

substituting

$$I = \frac{8}{7}\int \frac{\sqrt{7}/4 \, du}{u^2+1}$$

$$= \frac{2\sqrt{7}}{7}\int \frac{du}{u^2+1}$$

$$= \frac{2\sqrt{7}}{7}\tan^{-1}(u)+C$$

$$= \frac{2\sqrt{7}}{7}\tan^{-1}\left(\frac{4x+3}{\sqrt{7}}\right)+C$$

3 $I = \int \frac{dx}{\sqrt{3x^2+2x+1}}$

consider, $3x^2+2x+1 = 3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)$

$= 3\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)$

$I = \int \frac{dx}{\sqrt{3\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)}}$

$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x+\frac{1}{3}\right)^2+\left(\sqrt{\frac{2}{9}}\right)^2}}$

Let $u = x+\frac{1}{3}$

differentiate $u$ w.r.to $x$

$\frac{du}{dx} = 1$

$du = dx$

substituting,

$I = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2+\left(\frac{\sqrt{2}}{3}\right)^2}}$

$= \frac{1}{\sqrt{3}} \log\left|u+\sqrt{u^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right| + C$

$= \frac{1}{\sqrt{3}} \log\left|x+\frac{1}{3}+\sqrt{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right| + C$

$= \frac{1}{\sqrt{3}} \log\left|x+\frac{1}{3}+\sqrt{3x^2+2x+1}\right| + C$

4 $I = \int \frac{dx}{\sqrt{7+5x-3x^2}}$

consider, $7+5x-3x^2 = -3\left(x^2-\frac{5}{3}x\right)+7$

$$= -3\left(\left(x-\frac{5}{6}\right)^2-\frac{25}{36}\right)+7$$

$$= -3\left(x-\frac{5}{6}\right)^2+\frac{109}{12}$$

$$I = \int \frac{dx}{\sqrt{-3\left(x-\frac{5}{6}\right)^2+\frac{109}{12}}}$$

$$= \int \frac{dx}{\sqrt{3\left(\frac{109}{36}-\left(x-\frac{5}{6}\right)^2\right)}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{109}}{6}\right)^2-\left(x-\frac{5}{6}\right)^2}}$$

Let $u = x-\frac{5}{6}$

$du = dx$ (differentiate u w.r.t x)

$$I = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{\left(\frac{\sqrt{109}}{6}\right)^2-u^2}}$$

$$I = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{u}{\frac{\sqrt{109}}{6}}\right)+C$$

$$I = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{x-\frac{5}{6}}{\frac{\sqrt{109}}{6}}\right)+C$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6x-5}{\sqrt{109}}\right)+C$$

Q3

1 $I = \int x^2 \cos x \, dx$

$= x^2 \int \cos x \, dx - \int [d/dx(x^2) \int \cos x \, dx] dx$

$= x^2 \cdot \sin(x) - \int 2x \cdot \sin x \, dx$

$= x^2 \sin x - 2 \int x \sin x \, dx$

$\cancel{= x^2 \sin x - 2}$

~~consider $\int x \sin x \, dx = x \int \sin x \, dx - \int [d/dx(x) \int \sin x \, dx]$~~

consider $\int x \sin x = x \int \sin x \, dx - \int [d/dx(x) \int \sin x \, dx] dx$

$= \cancel{x} - \cos x - \int - \cos x \, dx$

$= -\cos x + \sin(x) + C.$

$\therefore I = x^2 \sin(x) + 2x \cos x - 2 \sin(x) + C.$

2 $I = \int x^2 \sin x$

$= x^2 \int \sin x \, dx - \int [d/dx(x^2) \cdot \int \sin x \, dx] dx$

$= x^2(-\cos x) - \int 2x(-\cos x) \, dx$

$= -x^2 \cos x + 2 \int x \cos x \, dx$

consider, $\int x \cos x = x \int \cos x - \int [d/dx(x) \cdot \int \cos x \, dx] dx$

$= x \cdot \sin x - \int \sin x \, dx$

$= x \sin x - (-\cos x) + C$

$= x \sin x + \cos x + C$

~~$\therefore I = x^2 \sin x +$~~

$I = -x^2 \cos x + 2(x \sin x + \cos x + C)$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$

$= 2 \cos x + 2x \sin x - x^2 \cos x + C.$

3 $I = \int x \log x \, dx$

$= \log x \int x \, dx - \int [d/dx(x) \cdot \int x \, dx] \, dx$

$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$

$= \frac{x^2 \log x}{2} - \frac{1}{2} \int x \, dx$

$= \frac{x^2 \log x}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + C$

4 $I = \int \log x \, dx$

$= \log x \int dx - \int [d/dx(\log x) \cdot \int dx] \, dx$

$= \log x \cdot x - \int \frac{1}{x} \cdot x \, dx$

$= x \log x - x + C$

$= x(\log x - 1) + C$

5 $I = \int \tan^{-1} x \, dx$

$= \tan^{-1} x \int dx - \int [d/dx(\tan^{-1} x) \cdot \int dx] \, dx$

$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$

$= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \log|1+x^2| + C$

6 $I = \int \sin^{-1}x \, dx$

$= \sin^{-1}x \int dx - \int \left[ \frac{d}{dx}(\sin^{-1}x) \cdot \int dx \right] dx$

$= \sin^{-1}x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x \, dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x + \log|\sqrt{1-x^2}| + C$

Q4

1 $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} \rightarrow (1)$

$\therefore I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} \text{ — } (2)$

Adding (1) & (2)

$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

$= \int_0^a dx$

$= [x]_0^a$

$= a - 0$

$2I = a$

$\boxed{I = \frac{a}{2}}$

(2) $I = \int_0^{\pi/2} \sin^2 x \, dx \rightarrow (1)$

$\therefore I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx$

$= \int_0^{\pi/2} \cos^2 x \, dx \rightarrow (2)$

Adding (1) & (2), we get

$2I = \int_0^{\pi/2} (\sin^2 x + \cos^2 x) \, dx$

$= \int_0^{\pi/2} 1 \, dx$

$= [x]_0^{\pi/2}$

$= \frac{\pi}{2} - 0$

$2I = \frac{\pi}{2}$

$\boxed{I = \frac{\pi}{4}}$

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3 $I = \int_0^{\pi/2} \log(\tan x)\, dx \rightarrow (1)$

$\therefore I = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$

$= \int_0^{\pi/2} \log(\cot x)\, dx$

Adding (1) & (2), we get

$2I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\, dx$

$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx$

$= \int_0^{\pi/2} 0\, dx$

$2I = 0$

$I = 0$

4 $I = \int_{-1}^{1} \frac{x^2}{1+x^2}\, dx$

$f(x) = \frac{x^2}{1+x^2}$

$f(-x) = \frac{(-x^2)^2}{1+(-x)^2} = \frac{x^2}{1+x^2} = f(x)$

As, $f(-x) = f(x)$, $f(x)$ is even function.

$\therefore\ I = 2\int_0^1 \frac{x^2}{1+x^2}\, dx$

$= 2\int_0^1 \frac{1+x^2-1}{1+x^2}\, dx$

$= 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$

$= 2\left[\int_0^1 dx - \int_0^1 \frac{dx}{1+x^2}\right]$

$= 2\left[[x]_0^1 - (\tan^{-1}x)_0^1\right]$

$= 2[(1-0) - (\tan^{-1}(1) - \tan^{-1}0)]$

$= 2\left[1 - \frac{\pi}{4}\right]$

$= 2 - \frac{\pi}{2}$

5 $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} \rightarrow (1)$

$$f(x) = \frac{\sin^4 x}{\sin^4 x + \cos^4 x}$$

$$f(-x) = \frac{\sin^4(-x)}{\sin^4(-x) + \cos^4(-x)}$$

$$= \frac{\sin^4(x)}{\sin^4(x) + \cos^4 x} = f(x)$$

$\therefore f(x)$ is even function.

$$I = 2\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx \longrightarrow (2)$$

$$= 2\int_0^{\pi/2} \frac{\sin^4(\pi/2 - x)}{\sin^4(\pi/2 - x) + \cos^4(\pi/2 - x)}\, dx$$

$$= 2\int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\, dx \rightarrow (3)$$

Adding (2) & (3).

$$2I = 2\int_0^{\pi/2} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x}\, dx$$

$$I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\boxed{I = \frac{\pi}{2}}$$

6 $I = \int_{-1}^{1} \frac{x^2 \sin x}{1+\cos x} dx \rightarrow (1)$

Here, $f(x) = \frac{x^2 \sin x}{1+\cos x}$

$f(-x) = \frac{(-x)^2 \sin(-x)}{1+\cos(-x)} = \frac{-x^2 \sin x}{1+\cos x} = -f(x)$

$\therefore f(x) = -f(x)$

$f(x)$ is odd function.

$\therefore I = 0$.

7 $I = \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}} dx \rightarrow (1)$

$\therefore I = \int_0^3 \frac{\sqrt{3-(3-x)}}{\sqrt{3-(3-x)}+\sqrt{3-x}} dx$

$= \int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}} dx \rightarrow (2)$

Adding (1) & (2):

$2I = \int_0^3 \frac{\sqrt{3-x}+\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} dx$

$2I = \int_0^3 dx$

$2I = [x]_0^3$

$2I = 3 - 0$

$\boxed{I = \frac{3}{2}}$

(8) $I = \int_4^5 \frac{\sqrt{5-x}}{\sqrt{x-4}+\sqrt{5-x}}\,dx \rightarrow (1)$

$\therefore I = \int_4^5 \frac{\sqrt{5-(9-x)}}{\sqrt{(9-x)-4}+\sqrt{5-(9-x)}}\,dx$

$= \int_4^5 \frac{\sqrt{x-4}}{\sqrt{5-x}+\sqrt{x-4}}\,dx \rightarrow (1)$

Adding (1) & (2)

$2I = \int_4^5 \frac{\sqrt{5-x}+\sqrt{x-4}}{\sqrt{5-x}+\sqrt{x-4}}\,dx$

$2I = \int_4^5 dx$

$2I = [x]_4^5$

$2I = 5-4$

$2I = 1$

$\boxed{I = \frac{1}{2}}$